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Spinor Casimir densities for a spherical shell in the global monopole spacetime

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Abstract
We investigate the vacuum expectation values of the energy–momentum tensor and the fermionic condensate associated with a massive spinor field obeying the MIT bag boundary condition on a spherical shell in the global monopole spacetime. In order to do that, we use the generalized Abel–Plana summation formula. As we shall see, this procedure allows us to extract from the vacuum expectation values the contribution coming from the unbounded spacetime and to explicitly present the boundary induced parts. As regards the boundary induced contribution, two distinct situations are examined: the vacuum average effects inside and outside the spherical shell. The asymptotic behaviour of the vacuum densities is investigated near the sphere centre and near the surface, and at large distances from the sphere. In the limit of strong gravitational field corresponding to small values of the parameter describing the solid angle deficit in the global monopole geometry, the sphere induced expectation values are exponentially suppressed. We discuss, as a special case, the fermionic vacuum densities for the spherical shell on the background of the Minkowski spacetime. Previous approaches to this problem within the framework of the QCD bag models have been global and our calculation is a local extension of these contributions.

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1. Introduction

Topological defects of different types [1] may have been formed during the phase transitions in the early universe. Depending on the topology of the vacuum manifold $\mathcal{M}$, these are domain walls, strings, monopoles and textures corresponding to the homotopy groups $\pi_0(\mathcal{M})$, $\pi_1(\mathcal{M})$,

$\pi_2(\mathcal{M})$ and $\pi_3(\mathcal{M})$, respectively. Physically, these topological defects appear as a consequence of the spontaneous breakdown of local or global gauge symmetries of the system composed of self-coupling scalar Higgs or Goldstone fields, respectively. Global monopoles are spherically symmetric topological defects created due to the phase transition when a global symmetry is spontaneously broken and they play an important role in cosmology and astrophysics.

The simplest theoretical model which provides global monopoles was proposed a few years ago by Barriola and Vilenkin [2]. This model is composed of a self-coupling iso-scalar Goldstone field triplet ϕ^a, whose original global $O(3)$ symmetry is spontaneously broken to $U(1)$. The matter field plays the role of an order parameter which, outside the monopole's core, acquires a non-vanishing value. The main part of the monopole's energy is concentrated in its small core. Coupling this system with the Einstein equations, a spherically symmetric metric tensor is found. Neglecting the small size of the monopole's core, this tensor can be approximately given by the line element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - \alpha^2 r^2 (\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2) \tag{1}$$

where the parameter α^2, smaller than unity, depends on the symmetry breaking energy scale and codifies the presence of the global monopole[4]. This spacetime corresponds to an idealized point-like global monopole. It is not flat: the scalar curvature $R = 2(1-\alpha^2)/r^2$, and the solid angle of a sphere of unit radius is $\Omega = 4\pi\alpha^2$, so smaller than the ordinary one. The energy–momentum tensor associated with this object has a diagonal form and its non-vanishing components read $T_0^0 = T_r^r = (\alpha^2 - 1)/r^2$.

The quantum effects due to the point-like global monopole spacetime on the matter fields have been considered in [3, 4] for massless scalar and fermionic fields, respectively. In order to do that, the scalar and spinor Green functions in this background were obtained. More recently, the effect of the temperature on these polarization effects was analysed in [5] for scalar and fermionic fields. The calculation of quantum effects on a massless scalar field in a higher dimensional global monopole spacetime has also been developed in [6].

Although the deficit solid angle and also the curvature associated with this manifold produce non-vanishing vacuum polarization effects on matter fields, the influence of boundary conditions obeyed by the matter fields on the vacuum polarization effects have been investigated. The Casimir energy associated with a massive scalar field inside a spherical region in the global monopole background have been analysed in [7, 8] using the zeta function regularization method. More recently, the Casimir densities induced by a single and two concentric spherical shells have been calculated [9, 10] for higher dimensional global monopole spacetime by making use of the generalized Abel–Plana summation formula [11, 12]. This procedure allows one to develop the summation over all discrete modes. Here we shall calculate the Casimir densities for fermionic fields obeying the MIT bag boundary condition on the spherical shell in the point-like global monopole spacetime. Specifically we shall calculate the renormalized vacuum expectation values of the energy–momentum tensor and the fermionic condensate in the regions inside and outside the spherical shell. As we shall see using the generalized Abel–Plana summation formula, all the components of the vacuum average of the energy–momentum tensor can be separated into two contributions: boundary dependent and independent ones. The boundary independent contribution is similar to the previous result obtained in [4] using a different approach. It is divergent and consequently, in order to obtain a finite and well defined expression, we must apply some regularization procedure. The boundary dependent contribution is finite at any strictly interior or exterior point and does not contain anomalies. Consequently, it does not require any regularization

[4] In fact the parameter $\alpha^2 = 1 - 8\pi G\eta^2$, with η being the energy scale where the global symmetry is spontaneously broken.

procedure. Because the analysis of the boundary independent term has been performed before, in this present analysis we shall concentrate on the boundary dependent part. Taking $\alpha = 1$, from our results in this paper we obtain, as a special case, the fermionic Casimir densities for a spherical shell on the background of the Minkowski spacetime. With the motivation of the MIT bag model in QCD, the corresponding Casimir effect was considered in a number of papers [13–21] (for reviews and additional references see [22–25]). To our knowledge, most of the previous studies were focused on global quantities, such as the total vacuum energy and stress on the surface. The density of the fermionic vacuum condensate for a massless spinor field inside the bag was investigated in references [15, 16] (see also [24]). In the considerations of the Casimir effect it is of physical interest to calculate not only the total energy but also the local characteristics of the vacuum, such as the energy–momentum tensor and vacuum condensates. In addition to describing the physical structure of the quantum field at a given point, the energy–momentum tensor acts as the source of gravity in the Einstein equations[5]. It therefore plays an important role in modelling a self-consistent dynamics involving the gravitational field [26]. For the case of the Minkowski bulk, our calculation is a local extension of the previous contributions on the fermionic Casimir effect for a spherical shell.

This paper is organized as follows. In section 2 we obtain the normalized eigenfunctions for a massive spinor field on the global monopole spacetime inside a spherical shell of finite radius. In section 3, using the generalized Abel–Plana summation formula, we formally obtain the vacuum expectation value of the energy–momentum tensor for the fermionic field obeying the MIT bag condition on the spherical shell. Explicit behaviour for the boundary dependent term is exhibited. Section 4 is devoted to the calculation of the vacuum expectation values for the region outside the shell. In section 5 we present our concluding remarks and leave to the appendix some relevant calculations.

2. The eigenfunctions for a spinor field on the global monopole spacetime

The dynamics of a massive spinor field on a curved spacetime is described by the Dirac equation

$$\mathrm{i}\gamma^{\mu}(\partial_{\mu} + \Gamma_{\mu})\psi - M\psi = 0 \tag{2}$$

where γ^{μ} are the Dirac matrices defined in such a curved spacetime, and Γ_{μ} is the spin connection defined as

$$\Gamma_{\mu} = \tfrac{1}{4}\gamma_{\nu}\nabla_{\mu}\gamma^{\nu} \tag{3}$$

∇_{μ} being the standard covariant derivative operator. Notice that, using the usual anticommutation relations for the Dirac matrices, $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$, we can see that

$$\gamma^{\mu}\Gamma_{\mu} = \tfrac{1}{4}\nabla_{\mu}\gamma^{\mu} + \tfrac{1}{8}\gamma^{\mu}\gamma^{\nu}(\partial_{\mu}\gamma_{\nu} - \partial_{\nu}\gamma_{\mu}). \tag{4}$$

After this brief introduction, let us now specialize to the spacetime associated with the point-like global monopole whose line element is described by (1). In order to develop such a procedure we shall adopt the following representation for the Dirac matrices:

$$\gamma^{0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^{k} = \begin{pmatrix} 0 & \sigma^{k} \\ -\sigma^{k} & 0 \end{pmatrix} \tag{5}$$

[5] The effects of the back-reaction corrections on the Einstein equation due to the vacuum polarization produced by a massless scalar field in a global monopole spacetime has been analysed in [3].

given in terms of the curved space Pauli 2 × 2 matrices σ^k. In the spherical coordinates corresponding to line element (1) the latter have the form

$$\sigma^1 = \begin{pmatrix} \cos\theta & \mathrm{e}^{-\mathrm{i}\phi}\sin\theta \\ \mathrm{e}^{\mathrm{i}\phi}\sin\theta & -\cos\theta \end{pmatrix} \qquad \sigma^2 = \frac{1}{\alpha r}\begin{pmatrix} -\sin\theta & \mathrm{e}^{-\mathrm{i}\phi}\cos\theta \\ \mathrm{e}^{\mathrm{i}\phi}\cos\theta & \sin\theta \end{pmatrix}$$
$$\sigma^3 = \frac{\mathrm{i}}{\alpha r\sin\theta}\begin{pmatrix} 0 & -\mathrm{e}^{-\mathrm{i}\phi} \\ \mathrm{e}^{\mathrm{i}\phi} & 0 \end{pmatrix}. \tag{6}$$

These matrices satisfy the relation

$$\sigma^l\sigma^k = \gamma^{lk} + \mathrm{i}\frac{\varepsilon^{lkm}}{\sqrt{\gamma}}\gamma_{mp}\sigma^p \tag{7}$$

where $\gamma^{lk} = -g^{lk}$ are the spatial components of the metric tensor and γ is the corresponding determinant. ε^{lkm} is the totally anti-symmetric symbol with $\varepsilon^{123} = 1$. Here and below, the italic indices $i, k, \ldots$ run over values 1, 2, 3. It can be easily checked that with these representations the Dirac matrices satisfy the standard anticommutation relations. Substituting these matrices into formula (4), we can see that

$$\gamma^\mu\Gamma_\mu = \frac{\alpha-1}{\alpha r}\gamma^1. \tag{8}$$

Let us write the four-component spinor field ψ in terms of two-component ones as

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}. \tag{9}$$

Assuming the time dependence in the form $\mathrm{e}^{-\mathrm{i}\omega t}$, from (2) one finds the equations for these spinors

$$\sigma^k\partial_k\varphi + \frac{\alpha-1}{\alpha r}(\hat{n}\cdot\vec{\sigma})\varphi = \mathrm{i}(\omega+M)\chi \tag{10a}$$

$$\sigma^k\partial_k\chi + \frac{\alpha-1}{\alpha r}(\hat{n}\cdot\vec{\sigma})\chi = \mathrm{i}(\omega-M)\varphi \tag{10b}$$

where $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^2)$ and $\hat{n} = \vec{r}/r$. The angular parts of the spinors are the standard spinor spherical harmonics Ω_{jlm} whose explicit form is given in [27]:

$$\varphi = f(r)\Omega_{jlm} \qquad \chi = (-1)^{(1+l-l')/2}g(r)\Omega_{jl'm} \tag{11}$$

where j specifies the value of the total angular momentum, and m its projection; $l = j \pm 1/2, l' = 2j - l$. Using the formula

$$\Omega_{jl'm} = i^{l-l'}(\hat{n}\cdot\vec{\sigma})\Omega_{jlm} \tag{12}$$

it can be seen that

$$\sigma^k\partial_k\varphi = \mathrm{i}^{l'-l}\left[f'(r) + \frac{1+\kappa}{\alpha r}f(r)\right]\Omega_{jl'm} \tag{13a}$$

$$\sigma^k\partial_k\chi = -\mathrm{i}\left[g'(r) + \frac{1-\kappa}{\alpha r}g(r)\right]\Omega_{jlm} \tag{13b}$$

where we use the notation

$$\kappa = \begin{cases} -(l+1) & j = l+1/2 \\ l & j = l-1/2. \end{cases} \tag{14}$$

By taking into account these relations, from (10) we obtain the following set of differential equations for the radial functions:

$$f'(r) + \frac{\alpha + \kappa}{\alpha r} f(r) - (\omega + M) g(r) = 0 \tag{15a}$$

$$g'(r) + \frac{\alpha - \kappa}{\alpha r} g(r) + (\omega - M) f(r) = 0. \tag{15b}$$

They lead to the second-order differential equations for the separate functions:

$$f''(r) + \frac{2}{r} f'(r) + \left[k^2 - \frac{\kappa(\kappa + \alpha)}{\alpha^2 r^2}\right] f(r) = 0 \qquad k = \sqrt{\omega^2 - M^2} \tag{16a}$$

$$g''(r) + \frac{2}{r} g'(r) + \left[k^2 - \frac{\kappa(\kappa - \alpha)}{\alpha^2 r^2}\right] g(r) = 0 \tag{16b}$$

with the solutions

$$f(r) = A \frac{Z_{|\kappa/\alpha + 1/2|}(kr)}{\sqrt{r}} \qquad g(r) = B \frac{Z_{|\kappa/\alpha - 1/2|}(kr)}{\sqrt{r}} \tag{17}$$

where $Z_\nu(x)$ represents the cylindrical Bessel function of the order ν. The constants A and B are related by equations (15)

$$B = \frac{\mp k A}{\omega + M} \qquad \text{for} \quad j = l \pm \frac{1}{2}. \tag{18}$$

As a result, for a given j we have eigenfunctions of two types with different parities corresponding to $j = l \pm 1/2$. These functions are specified by the set of quantum numbers $\beta = (\sigma k j l m)$ and have the form

$$\psi_\beta = \frac{A\,\mathrm{e}^{-\mathrm{i}\omega t}}{\sqrt{r}} \begin{pmatrix} Z_{\nu_\sigma}(kr)\Omega_{jlm} \\ \mathrm{i} n_\sigma Z_{\nu_\sigma + n_\sigma}(kr) \frac{k(\hat{n}\cdot\vec{\sigma})}{\omega + M} \Omega_{jlm} \end{pmatrix} \tag{19}$$

$$l = j - \frac{n_\sigma}{2} \qquad \omega = \pm E \qquad E = \sqrt{k^2 + M^2} \tag{20}$$

where $j = 1/2, 3/2, \ldots,$ and $m = -j, \ldots, j$,

$$\sigma = 0, 1 \qquad n_\sigma = (-1)^\sigma \qquad \nu_\sigma = \frac{j + 1/2}{\alpha} - \frac{n_\sigma}{2}. \tag{21}$$

On the basis of formula (19) we define the positive and negative frequency eigenfunctions as

$$\psi_\beta = \begin{cases} \psi_\beta^{(+)} & \text{for} \quad \omega > 0 \\ \psi_\beta^{(-)} & \text{for} \quad \omega < 0. \end{cases} \tag{22}$$

These functions are orthonormalized by the condition

$$\int \mathrm{d}^3 x \sqrt{\gamma}\, \psi_\beta^{(\eta)+} \psi_{\beta'}^{(\eta')} = \delta_{\eta\eta'} \delta_{\beta\beta'} \qquad \eta, \eta' = \pm \tag{23}$$

from which the normalization constant A can be determined.

3. Vacuum expectation values of the energy–momentum tensor inside a spherical shell

In this section we shall consider the vacuum expectation values of the energy–momentum tensor inside a spherical shell concentric with the global monopole. The integration in formula (23) goes over the interior region of the sphere and $Z_\nu(x) = J_\nu(x)$, where $J_\nu(x)$ is the Bessel function of the first kind. We shall assume that on the sphere surface the field satisfies bag boundary conditions:

$$(1 + \mathrm{i}\gamma^\mu n_{b\mu})\psi = 0 \qquad r = a \tag{24}$$

where a is the sphere radius, $n_{b\mu}$ is the outward-pointing normal to the boundary, for the sphere $n_{b\mu} = (0, 1, 0, 0)$. In terms of the spinors φ and χ this condition is written in the form

$$\varphi + \mathrm{i}(\vec{\sigma} \cdot \hat{n}_b)\chi = 0 \qquad r = a. \tag{25}$$

The imposition of this boundary condition on the eigenfunctions (19) leads to the following equations for the eigenvalues

$$J_{\nu_\sigma + n_\sigma}(ka) = n_\sigma \frac{\omega + M}{k} J_{\nu_\sigma}(ka). \tag{26}$$

This boundary condition can be written in the form

$$\tilde{J}_{\nu_\sigma}(ka) = 0 \tag{27}$$

where now and below for a given function $F(z)$ we shall use the notation

$$\tilde{F}(z) \equiv zF'(z) + \left(\mu + s_\omega\sqrt{z^2 + \mu^2} - (-1)^\sigma \nu\right)F(z) \qquad \sigma = 0, 1 \tag{28}$$

with $s_\omega = \mathrm{sgn}(\omega)$ and $\mu = Ma$. Let us denote by $\lambda_{\nu_\sigma,s} = ka$, $s = 1, 2, \ldots,$ the roots to equation (27) in the right half-plane, arranged in ascending order. By taking into account equation (26) and using the standard integral for the Bessel functions, from condition (23) for the normalization coefficient one finds

$$A = A_\sigma \qquad A_\sigma^{-2} \equiv \frac{2\alpha^2 a^2}{z^2} J_{\nu_\sigma}^2(z) \left[\left(a\omega - n_\sigma \frac{\nu_\sigma}{2}\right)^2 - \frac{\nu_\sigma^2}{4} - \frac{z^2}{2a(\omega + M)}\right] \qquad z = \lambda_{\nu_\sigma,s} \tag{29}$$

with $\omega = \pm\sqrt{\lambda_{l,s}^{(\sigma)2}/a^2 + M^2}$.

Now we expand the field operator in terms of the complete set of single-particle states $\left\{\psi_\beta^{(+)}, \psi_\beta^{(-)}\right\}$

$$\hat{\psi} = \sum_\beta \left(\hat{a}_\beta \psi_\beta^{(+)} + \hat{b}_\beta^+ \psi_\beta^{(-)}\right) \tag{30}$$

where $\hat{a}_\beta$ is the annihilation operator for particles and $\hat{b}_\beta^+$ is the creation operator for antiparticles. In order to find the vacuum expectation value for the operator of the energy–momentum tensor we substitute the expansion (30) and the analogous expansion for the operator $\hat{\bar{\psi}}$ into the corresponding expression for the spinor fields:

$$T_{\mu\nu}\{\hat{\bar{\psi}}, \hat{\psi}\} = \frac{\mathrm{i}}{2}[\hat{\bar{\psi}}\gamma_{(\mu}\nabla_{\nu)}\hat{\psi} - (\nabla_{(\mu}\hat{\bar{\psi}})\gamma_{\nu)}\hat{\psi}]. \tag{31}$$

By making use of the standard anticommutation relations for the annihilation and creation operators, for the vacuum expectation values one finds the following mode-sum formula:

$$\langle 0|T_{\mu\nu}|0\rangle = \sum_\beta T_{\mu\nu}\left\{\bar{\psi}_\beta^{(-)}(x), \psi_\beta^{(-)}(x)\right\} \tag{32}$$

where $|0\rangle$ is the amplitude for the corresponding vacuum. Since the spacetime is spherically symmetric and static, the vacuum energy–momentum tensor is diagonal; moreover, $\langle T_\theta^\theta\rangle = \langle T_\phi^\phi\rangle$. So in this case we can write

$$\langle 0|T_\mu^\nu|0\rangle = \mathrm{diag}(\varepsilon, -p, -p_\perp, -p_\perp) \tag{33}$$

in terms of the energy density ε and radial, p, and azimuthal, $p_\perp$, pressures. As a consequence of the continuity equation $\nabla_\nu\langle 0|T_\mu^\nu|0\rangle = 0$, these functions are related by the equation

$$r\frac{\mathrm{d}p}{\mathrm{d}r} + 2(p - p_\perp) = 0 \tag{34}$$

which means that the radial dependence of the radial pressure necessarily leads to the anisotropy in the vacuum stresses.

Substituting eigenfunctions (19) into equation (32), the summation over the quantum number m can be carried out by using the standard summation formula for the spherical harmonics. For the energy–momentum tensor components one finds

$$q(r) = \frac{-1}{8\pi\alpha^2 a^3 r}\sum_{j=1/2}^{\infty}(2j+1)\sum_{\sigma=0,1}\sum_{s=1}^{\infty} T_{\nu_\sigma}\left(\lambda_{\nu_\sigma,s}\right) f_{\sigma\nu_\sigma}^{(q)}\left[\lambda_{\nu_\sigma,s}, J_{\nu_\sigma}\left(\lambda_{\nu_\sigma,s} r/a\right)\right]$$
$$q = \varepsilon, p, p_\perp \tag{35}$$

where we have introduced the notation

$$f_{\sigma\nu}^{(\varepsilon)}[z, J_\nu(y)] = z\left[\left(\sqrt{z^2+\mu^2} - \mu\right)J_\nu^2(y) + \left(\sqrt{z^2+\mu^2} + \mu\right)J_{\nu+n_\sigma}^2(y)\right] \tag{36}$$

$$f_{\sigma\nu}^{(p)}[z, J_\nu(y)] = \frac{z^3}{\sqrt{z^2+\mu^2}}\left[J_\nu^2(y) - \frac{2\nu+n_\sigma}{y}J_\nu(y)J_{\nu+n_\sigma}(y) + J_{\nu+n_\sigma}^2(y)\right] \tag{37}$$

$$f_{\sigma\nu}^{(p_\perp)}[z, J_\nu(y)] = \frac{z^3(2\nu+n_\sigma)}{2y\sqrt{z^2+\mu^2}}J_\nu(y)J_{\nu+n_\sigma}(y). \tag{38}$$

Note that in (35) we have used the relation between the normalization coefficient and the function $T_\nu(z)$ introduced in the appendix

$$A_\sigma^2 = \frac{z}{2\alpha^2 a^2}\frac{\sqrt{z^2+a^2M^2} + aM}{\sqrt{z^2+a^2M^2}}T_{\nu_\sigma}(z) \qquad z = \lambda_{\nu_\sigma,s}. \tag{39}$$

The vacuum expectation values (35) are divergent and need some regularization procedure. To make them finite we can introduce a cut-off function $\Phi_\eta(z)$, $z = \lambda_{\nu_\sigma,s}$ with the cut-off parameter η, which decreases with increasing z and satisfies the condition $\Phi_\eta \to 1$, $\eta \to 0$. Now, to extract the boundary-free parts we apply to the corresponding sums over s the summation formula derived in the appendix. As a function $f(z)$ in this formula we take $f(z) = f_{\sigma\nu_\sigma}^{(q)}[z, J_{\nu_\sigma}(zr/a)]\Phi_\eta(z)$. As a result the components of the vacuum energy–momentum tensor can be presented in the form

$$q(r) = q_m(r) + q_b(r) \qquad q = \varepsilon, p, p_\perp \tag{40}$$

where the first term on the right-hand side comes from the integral on the left of summation formula (A.7) and the second term comes from the integral on the right of this formula. Making use of the asymptotic formulae for the Bessel modified functions, it can be seen that for $r < a$ the part $q_b(r)$ is finite in the limit $\eta \to 0$ and, hence, in this part the cut-off can be removed. As has been pointed out in the appendix, the function $f_{\sigma\nu_\sigma}^{(q)}[z, J_{\nu_\sigma}(zr/a)]$ satisfies relation

(A.10) and, hence, the part of the integral on the right of formula (A.7) over the interval $(0, \mu)$ vanishes after removing the cut-off. Introducing the notation

$$\nu \equiv \nu_1 = \frac{l}{\alpha} + \frac{1}{2} \tag{41}$$

explicitly summing over σ and transforming from summation over j to summation over $l = j + 1/2$, one obtains

$$q_m(r) = -\frac{1}{2\pi\alpha^2 r} \sum_{l=1}^{\infty} l \int_0^{\infty} \frac{x^3\,\mathrm{d}x}{\sqrt{x^2+M^2}} f_\nu^{(q)}[x, J_\nu(xr)] \tag{42}$$

where we use the notation

$$f_\nu^{(\varepsilon)}[x, J_\nu(y)] = \left(1 + \frac{M^2}{x^2}\right)\left[J_\nu^2(y) + J_{\nu-1}^2(y)\right] \tag{43}$$

$$f_\nu^{(p)}[x, J_\nu(y)] = J_\nu^2(y) + J_{\nu-1}^2(y) - \frac{2\nu}{y} J_\nu(y) J_{\nu-1}(y) \tag{44}$$

$$f_\nu^{(p_\perp)}[x, J_\nu(y)] = \frac{\nu}{y} J_\nu(y) J_{\nu-1}(y). \tag{45}$$

Introducing in the expression for $q_b(r)$ the modified Bessel functions, after some transformations and explicitly summing over σ, we obtain the formula

$$q_b(r) = \frac{1}{\pi^2\alpha^2 a^3 r} \sum_{l=1}^{\infty} l \int_\mu^{\infty} \frac{x^3\,\mathrm{d}x}{\sqrt{x^2-\mu^2}} \frac{W[I_\nu(x), K_\nu(x)]}{W[I_\nu(x), I_\nu(x)]} F_\nu^{(q)}[x, I_{\nu_\sigma}(xr/a)] \tag{46}$$

with

$$F_\nu^{(\varepsilon)}[x, I_\nu(y)] = \left(1 - \frac{\mu^2}{x^2}\right)\left\{I_{\nu-1}^2(y) - I_\nu^2(y) - \mu \frac{I_{\nu-1}^2(y) + I_\nu^2(y)}{W[I_\nu(x), K_\nu(x)]}\right\} \tag{47}$$

$$F_\nu^{(p)}[x, I_\nu(y)] = I_{\nu-1}^2(y) - I_\nu^2(y) - \frac{2\nu-1}{y} I_\nu(y) I_{\nu-1}(y) \tag{48}$$

$$F_\nu^{(p_\perp)}[x, I_\nu(y)] = \frac{\nu - 1/2}{y} I_\nu(y) I_{\nu-1}(y). \tag{49}$$

Here and below for given functions $f(x)$ and $g(x)$ we use the notation

$$W[f(x), g(x)] = [xf'(x) + (\mu+\nu) f(x)][xg'(x) + (\mu+\nu) g(x)] + (x^2 - \mu^2) f(x) g(x). \tag{50}$$

As we see, the part $q_m(r)$ in the vacuum expectation value for the energy–momentum tensor does not depend on the radius of the sphere a, whereas the contribution of the terms $q_b(r)$ tends to zero as $a \to \infty$ (for large a the subintegrand behaves as $e^{2x(r/a-1)}$). It follows from this that the quantities (42) are the vacuum expectation values for the components of the energy–momentum tensor for the unbounded global monopole space:

$$\langle 0_m | T_\mu^\nu | 0_m \rangle = \mathrm{diag}(\varepsilon_m, -p_m, -p_m, -p_{\perp m}) \tag{51}$$

where $|0_m\rangle$ is the amplitude for the corresponding vacuum. Note that in expressions (42) for the corresponding components we have not explicitly written down the cut-off function. To be precise, in this form all terms related to (42) are divergent. The renormalization prescriptions adopted to provide a finite and well defined result are the usual ones applied for the curved spacetime without a boundary [26, 28, 29]. For the specific system analysed here the point-splitting renormalization procedure has been applied in a previous publication [4]. The part $q_b(r)$ in equation (40) is induced by the presence of the spherical shell and can be termed the boundary part. As we have seen, the application of the generalized Abel–Plana formula allows us to extract from the vacuum expectation value of the energy–momentum tensor the contribution due to the boundary-free monopole spacetime and to present the boundary induced part in terms of exponentially convergent integrals (for applications of the generalized Abel–Plana formula to a number of Casimir problems with various boundary geometries, see [11, 12, 30–34]). It can be easily checked that both the terms on the right of formula (40), $q_m(r)$ and $q_b(r)$, obey the continuity equation (34). In addition, as is seen from expressions (46)–(49), for a massless spinor field the boundary induced part of the vacuum energy–momentum tensor is traceless and the trace anomalies are contained only in the purely global monopole part without boundaries.

Having the components of the energy–momentum tensor, we can find the corresponding fermionic condensate $\langle 0|\bar{\psi}\psi|0\rangle$ making use of the formula for the trace of the energy–momentum tensor, $T_\mu^\mu = M\bar{\psi}\psi$. It is presented in the form of the sum

$$\langle 0|\bar{\psi}\psi|0\rangle = \langle 0_m|\bar{\psi}\psi|0_m\rangle + \langle\bar{\psi}\psi\rangle_b \tag{52}$$

where the boundary-free part (first summand on the right) and the sphere induced part (second summand on the right) are determined by formulae similar to equations (42) and (46), respectively, with the replacements

$$f_\nu^{(q)}[x, J_\nu(y)] \to \frac{M}{x^2}\left[J_\nu^2(y) + J_{\nu-1}^2(y)\right] \tag{53}$$

$$F_\nu^{(q)}[x, I_\nu(y)] \to -\frac{a}{x^2}\left\{\mu\left[I_{\nu-1}^2(y) - I_\nu^2(y)\right] + (x^2-\mu^2)\frac{I_{\nu-1}^2(y) + I_\nu^2(y)}{W[I_\nu(x), K_\nu(x)]}\right\} \tag{54}$$

with the notation (50). Alternatively one could obtain formulae (52)–(54) by applying the summation formula (A.7) to the corresponding mode-sum $\sum_\beta \bar{\psi}_\beta^{(-)}\psi_\beta^{(-)}$ for the fermionic condensate.

Note that formulae (40), (42), (46) can be obtained in another equivalent way, applying a certain first-order differential operator to the corresponding Green function and taking the coincidence limit. To construct the Green function we can use the corresponding mode expansion formula with eigenfunctions (19). This function is a 4×4 matrix and the angular parts of the corresponding elements are products of the components for the spinor spherical harmonics, $\Omega_{jlm}^{(n)}(\theta,\phi)\Omega_{jlm}^{(n')+}(\theta',\phi')$, where the upper indices $n, n' = 1, 2$ enumerate the spinor components. These parts are the same as in the boundary-free case and coincide with the corresponding functions for the Minkowski bulk. The radial parts for the components of the Green function contain the products of the Bessel functions in the forms $J_{\nu_\sigma+\tau n_\sigma}(zr/a)J_{\nu_\sigma+\tau' n_\sigma}(zr'/a)$, $\tau, \tau' = 0, 1$, where $z = \lambda_{\nu_\sigma,s}$. To evaluate the sum over s we can apply the summation formula (A.7). The condition (A.4) is satisfied if $r + r' + |t - t'| < 2a$. The term with the integral on the left in formula (A.7) gives the Green function for the boundary-free global monopole spacetime, and the term with the integral on the right will give the boundary induced part.

In the case $\alpha = 1$ the quantities (42) present the vacuum expectation values for the Minkowski spacetime without boundaries. This can be also seen by direct evaluation. For example, in the case of the energy density, making use of the formula $\sum_{l=0}^\infty (2l+1)J_{l+1/2}^2(y) = 2y/\pi$, one finds

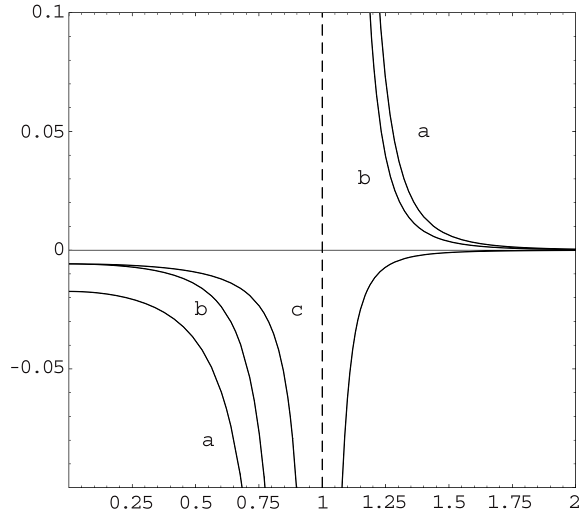

Figure 1. The vacuum energy density, $a^4\varepsilon$ (curve a), azimuthal pressure $a^4 p_\perp$ (curve b) and radial pressure $a^4 p$ (curve c) for a massless spinor as functions of the ratio r/a inside and outside a spherical shell in the Minkowski spacetime ($\alpha = 1$).

$$\begin{aligned}\varepsilon_m(r) &= -\frac{1}{2\pi r}\sum_{l=0}^{\infty} l\int_0^\infty \mathrm{d}x\, x^2\sqrt{x^2+M^2}\big[J^2_{l+1/2}(xr)+J^2_{l-1/2}(xr)\big]\\ &= -2\int\frac{\mathrm{d}^3k}{(2\pi)^3}\sqrt{k^2+M^2}\end{aligned} \tag{55}$$

which is precisely the energy density of the Minkowski vacuum for a spinor field. As for the Minkowski background, the renormalized vacuum energy–momentum tensor vanishes, $q_m(r)_{\mathrm{ren}} = 0$, the vacuum energy–momentum tensor is purely boundary induced and the corresponding components are given by formulae (46)–(49) with $\nu = l + 1/2$. Note that the previous investigations on the spinor Casimir effect for a spherical boundary (see, for instance, [13, 14, 17–25] and references therein) consider mainly global quantities, such as the total vacuum energy. For the case of a massless spinor the density of the fermionic condensate $\langle\bar\psi\psi\rangle_b$ is investigated in [15, 16] (see also [24]). The corresponding formula derived in [15] is obtained from (46) with the replacement (54) in the limit $\mu = 0$. In figure 1 we have presented the dependence of the Casimir densities, $a^4 q_b(r)$, on the rescaled radial coordinate r/a for a massless spinor field on the Minkowski bulk. The vacuum energy density and pressures are negative inside the sphere.

Now we turn to the consideration of various limiting cases of the expressions for the sphere induced vacuum expectation values. In the limit $r \to 0$, for the boundary parts (46) the summands with a given l behave as $r^{2l/\alpha-2}$, and the leading contributions come from the lowest $l = 1$ terms. Making use of standard formulae for the Bessel modified functions for small values of the argument, for the sphere induced parts near the centre, $r \ll a$, one finds

$$\varepsilon_b \approx \frac{\pi^{-2}a^{-4}}{2\alpha^2\Gamma^2\left(\frac{1}{\alpha}+\frac{1}{2}\right)}\left(\frac{r}{2a}\right)^{\frac{2}{\alpha}-2}\int_\mu^\infty \mathrm{d}x x^{\frac{2}{\alpha}}\sqrt{x^2-\mu^2}\frac{W[I_\nu(x),K_\nu(x)]-\mu}{W[I_\nu(x),I_\nu(x)]} \tag{56}$$

$$p_b \approx \alpha p_{\perp b} \approx \frac{\pi^{-2}a^{-4}}{2\alpha(2+\alpha)\Gamma^2\left(\frac{1}{\alpha}+\frac{1}{2}\right)}\left(\frac{r}{2a}\right)^{\frac{2}{\alpha}-2}\int_\mu^\infty \frac{x^{\frac{2}{\alpha}+2}\,\mathrm{d}x}{\sqrt{x^2-\mu^2}}\frac{W[I_\nu(x),K_\nu(x)]}{W[I_\nu(x),I_\nu(x)]} \tag{57}$$

where $\nu = 1/\alpha + 1/2$ and $\Gamma(x)$ is the gamma function. Hence, at the sphere centre the boundary parts vanish for the global monopole spacetime ($\alpha < 1$) and are finite for the Minkowski spacetime ($\alpha = 1$). Note that in the large mass limit, $\mu \gg 1$, the integrals in equations (56), (57) are exponentially suppressed by the factor $e^{-2\mu}$. In the Minkowski background case the vacuum stresses for a massless spinor are isotropic at the sphere centre and after the numerical evaluation of the integral one finds

$$p_b(0) = p_{\perp b}(0) = \frac{\varepsilon_b}{3} = \frac{2}{3\pi^2 a^4} \int_0^\infty \frac{\mathrm{e}^{-2x}(x^2 + x - \mathrm{e}^x \sinh x)\,\mathrm{d}x}{(2x^2+1)\cosh(2x) - 2x\sinh(2x) - 1} = -\frac{0.005\,79}{a^4}. \tag{58}$$

The boundary induced parts of the vacuum energy–momentum tensor components diverge at the sphere surface, $r \to a$. These divergences are well known in quantum field theory with boundaries and have been investigated for boundary geometries of various types [35–37]. In order to find the leading terms of the corresponding asymptotic expansion in powers of the distance from the sphere surface, we note that in the limit $r \to a$ the sum over l in (46) diverges and, hence, for small $1 - r/a$ the main contribution comes from the large values of l. Consequently, rescaling the integration variable $x \to \nu x$ and making use of the uniform asymptotic expansions for the modified Bessel functions for large values of the order [38], to the leading order one finds

$$\varepsilon_b(r) \approx -\frac{\mu + 1/5}{12\pi^2 a(a-r)^3} \tag{59}$$

$$p_b(r) \approx \left(1 - \frac{r}{a}\right) p_{\perp b}(r) \approx -\frac{1/5 - 2\mu}{24\pi^2 a^2 (a-r)^2}. \tag{60}$$

Notice that the terms in these expansions diverging as the inverse fourth power of the distance have cancelled out. This is a consequence of the conformal invariance of the massless fermionic field and is in agreement with the general conclusions of reference [36]. Near the sphere surface the energy density is negative for all values of μ, while the vacuum pressures are negative for $\mu < 0.1$ and are positive for $\mu > 0.1$. It is of interest to note that the leading terms do not depend on the parameter α and, hence, are the same for the Minkowski and global monopole bulks. For the latter case, due to the divergences, near the sphere surface the total vacuum energy–momentum tensor is dominated by the boundary induced parts $q_b(r)$. The dependence of these parts on the rescaled radial coordinate r/a is depicted in figure 2 for the case of a massless fermionic field on the global monopole background with the solid angle deficit parameter $\alpha = 0.5$.

Now let us consider the limit $\alpha \ll 1$ for a fixed value $r < a$. This limit corresponds to strong gravitational fields. In this case, from (41) one has $\nu \approx l/\alpha \gg 1$ and, after introducing in (46) a new integration variable $y = x/\nu$, we can replace the modified Bessel functions by their uniform asymptotic expansions for large values of the order. The integral over y can be estimated by making use of the Laplace method. The main contribution to the sum over l comes from the summands with $l = 1$ and the boundary parts of the vacuum energy–momentum tensor components behave as $\exp[-2\ln(a/r)/\alpha]$ with $p_b/p_{\perp b} \sim \alpha$. Hence, for $\alpha \ll 1$ the boundary induced vacuum expectation values are exponentially suppressed and the corresponding vacuum stresses are strongly anisotropic. Figure 3 shows that the nonzero mass can essentially change the behaviour of the vacuum energy–momentum tensor components. In this figure we have depicted the dependence of the boundary induced quantities $a^4 q_b(r)$ on the parameter Ma for the radial coordinate $r = 0.5a$. The left panel corresponds to the sphere in the Minkowski spacetime ($\alpha = 1$) and for the right panel $\alpha = 0.5$.

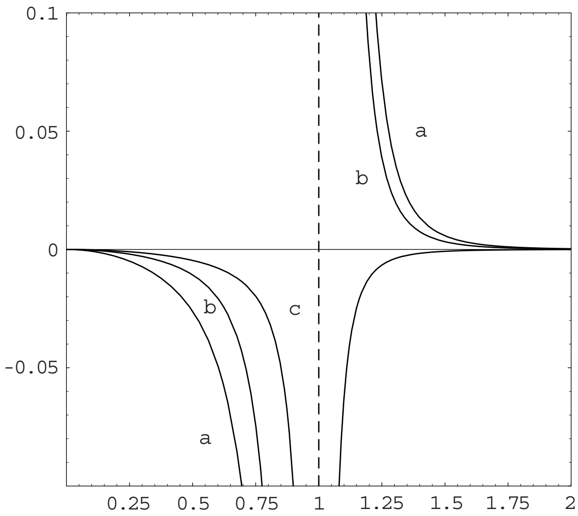

Figure 2. As figure 1, but for the boundary induced parts $a^4 q_b(r)$ on the background of the global monopole spacetime with $\alpha = 0.5$.

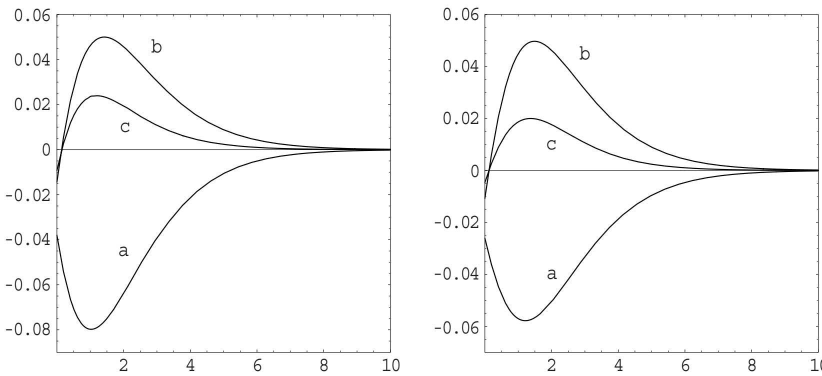

Figure 3. Boundary induced vacuum expectation values $a^4 q_b(r)$, $q = \varepsilon, p, p_\perp$, as functions of $\mu = Ma$ for $r/a = 0.5$. The curves a, b, c correspond to the energy density (ε), azimuthal pressure ($p_\perp$) and radial pressure (p), respectively. For the left panel $\alpha = 1$ (Minkowski spacetime) and for the right panel $\alpha = 0.5$.

4. Vacuum expectation values outside a spherical shell

Now let us consider the expectation values of the energy–momentum tensor in the region outside a spherical shell, $r > a$. The corresponding eigenfunctions have the form (19), where now the function $Z_\nu(kr)$ is a linear combination of the Bessel functions of the first and second kinds. The coefficient in this linear combination is determined from the boundary condition (25) and one obtains

$$Z_\nu(kr) = g_\nu(ka, kr) \equiv J_\nu(kr)\tilde{Y}_\nu(ka) - Y_\nu(kr)\tilde{J}_\nu(ka) \tag{61}$$

where $Y_\nu(z)$ is the Bessel function of the second kind, and the functions with tildes are defined as (28). Now the spectrum for the quantum number k is continuous and the corresponding $\delta_{kk'}$ in equation (23) is understood as the Dirac delta function $\delta(k - k')$. To find the normalization coefficient A from equation (23) it is convenient to take $\beta = \beta'$ for all discrete quantum numbers. As the normalization integral diverges in the limit $k = k'$, the main contribution to the integral over the radial coordinate comes from large values of r when the Bessel functions can be replaced by their asymptotic forms for large arguments. The resulting integral is elementary and for the normalization coefficient we obtain

$$A = A_\sigma \qquad A_\sigma^2 = \frac{k(\omega + M)}{2\alpha^2\omega\left[\tilde{J}^2_{\nu_\sigma}(ka) + \tilde{Y}^2_{\nu_\sigma}(ka)\right]}. \tag{62}$$

Substituting the eigenfunctions (19) into the mode-sum formula (32) and taking into account equations (61) and (62), we can see that the vacuum energy–momentum tensor has the form (33). The diagonal components are determined by the formulae

$$q(r) = \frac{-1}{8\pi\alpha^2 a^3 r}\sum_{j=1/2}^{\infty}(2j+1)\sum_{\sigma=0,1}\int_0^\infty \mathrm{d}x\,\frac{f^{(q)}_{\sigma\nu_\sigma}[x, g_{\nu_\sigma}(x, xr/a)]}{\tilde{J}^2_{\nu_\sigma}(x) + \tilde{Y}^2_{\nu_\sigma}(x)} \qquad q = \varepsilon, p, p_\perp \tag{63}$$

where the expressions for $f^{(q)}_{\sigma\nu_\sigma}\left[x, g_{\nu_\sigma}(x, xr/a)\right]$ are obtained from formulae (36)–(38) by making the replacements

$$J_\nu(y) \to g_\nu(x, y) \qquad J_{\nu+n_\sigma}(y) \to J_{\nu+n_\sigma}(y)\tilde{Y}_\nu(ka) - Y_{\nu+n_\sigma}(y)\tilde{J}_\nu(ka). \tag{64}$$

To find the parts in the vacuum expectation values of the energy–momentum tensor induced by the presence of the sphere, we subtract the corresponding components for the monopole bulk without boundaries, given by equation (42). In order to evaluate the corresponding difference we use the relation

$$\frac{f^{(q)}_{\sigma\nu_\sigma}\left[x, g_{\nu_\sigma}(x, xr/a)\right]}{\tilde{J}^2_{\nu_\sigma}(x) + \tilde{Y}^2_{\nu_\sigma}(x)} - f^{(q)}_{\sigma\nu_\sigma}\left[x, J_{\nu_\sigma}(xr/a)\right] = -\frac{1}{2}\sum_{s=1,2}\frac{\tilde{J}_{\nu_\sigma}(x)}{\tilde{H}^{(s)}_{\nu_\sigma}(x)} f^{(q)}_{\sigma\nu_\sigma}\left[x, H^{(s)}_{\nu_\sigma}(xr/a)\right] \tag{65}$$

where $H^{(s)}_\nu(z)$, $s = 1, 2$ are the Hankel functions. This allows us to present the vacuum energy–momentum tensor components in the form (40) with the boundary induced parts

$$q_b(r) = \frac{1}{16\pi\alpha^2 a^3 r}\sum_{j=1/2}^{\infty}(2j+1)\sum_{\sigma=0,1}\sum_{s=1,2}\int_0^\infty \mathrm{d}x\,\frac{\tilde{J}_{\nu_\sigma}(x)}{\tilde{H}^{(s)}_{\nu_\sigma}(x)} f^{(q)}_{\sigma\nu_\sigma}\left[x, H^{(s)}_{\nu_\sigma}(xr/a)\right]. \tag{66}$$

In the complex plane x we can rotate the integration contour on the right of this formula by the angle $\pi/2$ for $s = 1$ and by the angle $-\pi/2$ for $s = 2$. The integrals over the segments $(0, i\mu)$ and $(0, -i\mu)$ cancel out and, after introducing the Bessel modified functions, one obtains

$$q_b(r) = \frac{1}{\pi^2\alpha^2 a^3 r}\sum_{l=1}^{\infty} l\int_\mu^\infty \frac{x^3\,\mathrm{d}x}{\sqrt{x^2-\mu^2}}\frac{W[I_\nu(x), K_\nu(x)]}{W[K_\nu(x), K_\nu(x)]}F^{(q)}_\nu[x, K_\nu(xr/a)]. \tag{67}$$

Here the expressions for the functions $F^{(q)}_{\sigma\nu}[x, K_\nu(y)]$ are obtained from formulae (47)–(49) by making the replacements $I_\nu(y) \to K_\nu(y)$ and $I_{\nu-1}(y) \to -K_{\nu-1}(y)$. With the same replacements, from (52), (53), (54) we can obtain formulae for the fermionic condensate in the region outside a sphere. As for the interior region, it can be seen that in the limit of strong gravitational fields, $\alpha \ll 1$, the boundary induced vacuum expectation values are exponentially suppressed by the factor $\exp[-(2/\alpha)\ln(r/a)]$, and the corresponding vacuum stresses are strongly anisotropic: $p_b/p_{\perp b} \sim \alpha$.

In the case $\alpha = 1$ the renormalized values for the boundary-free parts $q_m(r)$ vanish and from (67) with $\nu = l + 1/2$ the components of the vacuum energy–momentum tensor are obtained for the region outside a spherical shell on the Minkowski bulk. Previous approaches to this problem have been global and our calculation is a local extension of these results. Note that for an infinitely thin spherical shell the total vacuum energy for a massless spinor, including interior and exterior parts, is positive, $E = 0.0204/a$ [17, 20, 21]. In figure 1 we have plotted the dependence of the vacuum energy density and stresses on the radial coordinate for a massless spinor field outside a sphere on the Minkowski bulk. The same graphical representations for the boundary induced expectation values (67) on the background of the global monopole spacetime with $\alpha = 0.5$ are depicted in figure 2. As seen from these figures, the energy density and azimuthal pressure are positive outside a sphere, and the radial pressure is negative. The latter has the same sign as for the interior region.

For the case of a massless spinor the asymptotic behaviour of the boundary part (67) at large distances from the sphere can be obtained by introducing a new integration variable $y = xr/a$ and expanding the subintegrands in terms of a/r. The leading contribution for the summands with a given l has the order $(a/r)^{2\nu+4}$ and the main contribution comes from the $l = 1$ term. Evaluating the standard integrals involving the square of the MacDonald function, the leading terms for the asymptotic expansions over a/r can be presented in the form

$$q_b(r) \approx \frac{1}{2^{\frac{2}{\alpha}}\pi a^4} \frac{\Gamma\left(\frac{1}{\alpha}+1\right)\Gamma\left(\frac{2}{\alpha}+\frac{3}{2}\right) f_q}{(4-\alpha^2)(2+\alpha)\Gamma^3\left(\frac{1}{\alpha}+\frac{1}{2}\right)} \left(\frac{a}{r}\right)^{\frac{2}{\alpha}+5} \tag{68}$$

where

$$f_\varepsilon = 4\frac{\alpha+1}{3\alpha+2} \qquad f_p = -\frac{2\alpha}{3\alpha+2} \quad f_{p_\perp} = 1. \tag{69}$$

As for the interior components, the quantities (67) diverge at the sphere surface $r = a$. Near the surface the dominant contributions come from modes with large l and, by making use of the uniform asymptotic expansions for the Bessel modified functions, the asymptotic expansions can be derived in powers of the distance from the sphere. The leading terms of these asymptotic expansions are determined by the formulae

$$\varepsilon_b(r) \sim \frac{1/5 - 5\mu}{12\pi^2 a(r-a)^3} \tag{70}$$

$$p_b(r) \sim -\left(\frac{r}{a} - 1\right) p_{\perp b}(r) \sim -\frac{1/5 - 2\mu}{24\pi^2 a^2 (r-a)^2}. \tag{71}$$

Near the sphere the external energy density is positive for $\mu < 0.04$ and is negative for $\mu > 0.04$. Recall that near the sphere the interior energy density is always negative. As we see, the leading terms for the radial pressure are the same for the regions outside and inside the sphere. For the azimuthal pressure these terms have opposite signs. In the case of the massless spinor field the same is true for the energy density.

5. Concluding remarks

In this paper we have analysed the fermionic Casimir densities induced by a spherical shell in an idealized point-like global monopole spacetime. Specifically, the renormalized vacuum expectation value of the energy–momentum tensor operator has been considered, where the matter fields obey the MIT bag boundary condition on the shell. Because the boundary condition provides a discrete energy spectrum for the matter fields, the summation over the modes can be developed by using the generalized Abel–Plana summation formula. Moreover, this procedure allows us to extract from the vacuum average the boundary dependent part. This part presents, besides the contribution coming from the parameter α which characterizes the presence of the global monopole, contributions coming from the boundary itself. Two distinct situations have been considered: the calculation of the vacuum average in the regions inside and outside of the spherical shell. As regards the interior region case, we pointed out that all contributions to the vacuum average go to zero as the radius of the spherical shell goes to infinity, as was expected.

The boundary induced expectation values for the components of the energy–momentum tensor are given by formulae (46) and (67) for interior and exterior regions, respectively. The corresponding formulae for the fermionic condensate densities are obtained from these expressions with the replacement (54) for the interior region and with additional replacement $I_\nu(y) \to K_\nu(y)$, $I_{\nu-1}(y) \to -K_{\nu-1}(y)$ for the exterior region. These expressions diverge in a non-integrable manner as the boundary is approached. The energy density and azimuthal pressure vary to leading order, as the inverse cube of the distance from the sphere, and near the sphere the azimuthal pressure has opposite signs for the interior and exterior regions. For a massless spinor the same is true for the energy density. The radial pressure varies as the inverse square of the distance and near the sphere has the same sign for exterior and interior regions. This behaviour is clearly seen from figures 1 and 2 where the radial dependences of the vacuum energy density and azimuthal and radial pressures are presented for the Minkowski background and global monopole spacetime with $\alpha = 0.5$. The leading terms of the corresponding asymptotic expansions near the sphere do not depend on the solid angle deficit parameter and are the same for these two cases. Near the sphere the interior energy density is negative for all values of the mass, while the exterior energy density is positive for $Ma < 0.04$ and is negative for $Ma > 0.04$. In order to illustrate the dependence on the mass, in figure 3 we have plotted the boundary induced vacuum densities at $r = 0.5a$ as functions of Ma. Near the sphere centre the dominant contributions come from modes with $l = 1$, and the sphere induced vacuum expectation values vanish for the global monopole spacetime and are finite for the Minkowski bulk. The asymptotic behaviour of the vacuum energy–momentum tensor at large distances from the sphere is described by formula (68) and the corresponding diagonal components go to zero as $(a/r)^{2/\alpha+5}$. In the limit of strong gravitational field, corresponding to small values of the parameter α, describing the solid angle deficit, the boundary induced part of the vacuum energy–momentum tensor is strongly suppressed by the factor $\exp[-(2/\alpha)|\ln(r/a)|]$ and the corresponding vacuum stresses are strongly anisotropic: $p_b \sim \alpha p_{\perp b}$. Note that this suppression effect also takes place in the scalar case [9, 10].

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Appendix. The formula for summation over the zeros of a combination of the Bessel functions

We have seen that the vacuum expectation values for the energy–momentum tensor for a spinor field inside a spherical shell on the background of the global monopole spacetime contain sums over the zeros of the function

$$\tilde{J}_\nu(z) \equiv zJ'_\nu(z) + \left(\mu + s_\omega\sqrt{z^2+\mu^2} - (-1)^\sigma \nu\right)J_\nu(z) \qquad \sigma = 0, 1 \quad s_\omega = \pm 1. \tag{A.1}$$

To obtain a formula for summation over these zeros, we use here the generalized Abel–Plana formula [11, 12]. In this formula, as the function $g(z)$ let us choose

$$g(z) = \mathrm{i}\frac{\tilde{Y}_\nu(z)}{\tilde{J}_\nu(z)} f(z) \tag{A.2}$$

with a function $f(z)$ analytic in the right half-plane $\mathrm{Re}\, z \geqslant 0$, and $Y_\nu(z)$ the Neumann function. For the sum and difference of the functions $f(z)$ and $g(z)$ one obtains

$$f(z) - (-1)^k g(z) = \frac{\tilde{H}_\nu^{(k)}(z)}{\tilde{J}_\nu(z)} f(z) \qquad k = 1, 2 \tag{A.3}$$

with $H_\nu^{(1)}$ and $H_\nu^{(2)}$ being Bessel functions of the third kind or Hankel functions. By using the asymptotic formulae for the Bessel functions for large values of the argument, the conditions for the generalized Abel–Plana formula can be written in terms of the function $f(z)$ as follows:

$$|f(z)| < \epsilon(x)\,\mathrm{e}^{c|y|} \qquad z = x + \mathrm{i}y \quad |z| \to \infty \tag{A.4}$$

where $c < 2$ and $\epsilon(x) \to 0$ for $x \to \infty$.

Let $\lambda_{\nu,s} \neq 0, s = 1, 2, 3 \ldots,$ be zeros for the function $\tilde{J}_\nu(z)$ in the right half-plane, arranged in ascending order, $\lambda_{\nu,s} \leqslant \lambda_{\nu,s+1}$. By using the Wronskian $W[J_\nu(z), Y_\nu(z)] = 2/\pi z$, one can easily see that $\tilde{Y}(\lambda_{\nu,s}) = 2/(\pi \tilde{J}(\lambda_{\nu,s}))$. This allows one to present the residue term coming from the poles of the function $g(z)$ in the form

$$\pi \mathrm{i}\, \mathrm{Res}_{z=\lambda_{\nu,s}} g(z) = T_\nu(\lambda_{\nu,s}) f(\lambda_{\nu,s}) \tag{A.5}$$

where we have introduced the notation

$$T_\nu(z) = \frac{z}{J_\nu^2(z)\left[z^2 + (\mu - (-1)^\sigma \nu)\left(\mu^2 + s_\omega\sqrt{z^2+\mu^2}\right) - \frac{s_\omega z^2}{2\sqrt{z^2+\mu^2}}\right]}. \tag{A.6}$$

Substituting (A.2) and (A.3) into the generalized Abel–Plana formula [11, 12] and taking in this formula the limit $a \to 0$ (the branch points $z = \pm \mathrm{i}\mu$ are avoided by semicircles of small radius), we obtain that for the function $f(z)$ analytic in the half-plane $\mathrm{Re}\, z > 0$ and satisfying condition (A.4) the following formula applies:

$$\lim_{b\to+\infty}\left\{\sum_{s=1}^{n} T_\nu(\lambda_{\nu,s}) f(\lambda_{\nu,s}) - \int_0^b f(x)\,\mathrm{d}x\right\} = \frac{\pi}{2}\mathrm{Res}_{z=0} f(z)\frac{\bar{Y}_\nu(z)}{\bar{J}_\nu(z)}$$
$$- \frac{1}{\pi}\int_0^\infty \left[\mathrm{e}^{-\nu\pi\mathrm{i}} f(x\mathrm{e}^{\pi\mathrm{i}/2})\frac{K_\nu^{(+)}(x)}{I_\nu^{(+)}(x)} + \mathrm{e}^{\nu\pi\mathrm{i}} f(x\mathrm{e}^{-\pi\mathrm{i}/2})\frac{K_\nu^{(-)}(x)}{I_\nu^{(-)}(x)}\right]\mathrm{d}x \tag{A.7}$$

where on the left $\lambda_{\nu,n} < b < \lambda_{\nu,n+1}$, and $T_\nu(\lambda_{\nu,k})$ is determined by relation (A.6). In formula (A.7) we use the notation

$$F^{(\pm)}(z) = \begin{cases} zF'(z) + \left(\mu + s_\omega\sqrt{\mu^2 - z^2} - (-1)^\sigma \nu\right)F(z) & |z| < \mu \\ zF'(z) + \left(\mu \pm s_\omega \mathrm{i}\sqrt{z^2 - \mu^2} - (-1)^\sigma \nu\right)F(z) & |z| > \mu \end{cases} \tag{A.8}$$

for a given function $F(z)$, and have taken into account that for $z = iy$,

$$(z^2 + \mu^2)^{1/2} = \begin{cases} (\mu^2 - y^2)^{1/2} & |y| < \mu \\ (y^2 - \mu^2)^{1/2}\,\mathrm{e}^{\mathrm{i}\pi/2} & y > \mu \\ (y^2 - \mu^2)^{1/2}\,\mathrm{e}^{-\mathrm{i}\pi/2} & y < -\mu \end{cases} \tag{A.9}$$

In this paper, we apply formula (A.7) to the sums over s in expressions (35) for the vacuum expectation values of the energy density and vacuum stresses. As can be seen from expressions (36)–(38), the corresponding functions $f(z)$ satisfy the relation

$$\mathrm{e}^{-\nu\pi\mathrm{i}} f(x\mathrm{e}^{\pi\mathrm{i}/2}) = -\mathrm{e}^{\nu\pi\mathrm{i}} f(x\mathrm{e}^{-\pi\mathrm{i}/2}) \qquad \text{for} \quad 0 \leqslant x < \mu. \tag{A.10}$$

By taking into account that for these x one has $F^{(+)} = F^{(-)}$, we conclude that the part of the integral on the right of equation (A.7) over the interval $(0, \mu)$ vanishes.

References

[1] Vilenkin A and Shellard E P S 1994 *Cosmic Strings and Other Topological Defects* (Cambridge: Cambridge University Press)
[2] Barriola M and Vilenkin A 1989 *Phys. Rev. Lett.* **63** 341
[3] Mazzitelli F D and Lousto C O 1991 *Phys. Rev.* D **43** 468
[4] Bezerra de Mello E R, Bezerra V B and Khusnutdinov N R 1999 *Phys. Rev.* D **60** 063506
[5] Carvalho F C and Bezerra de Mello E R 2001 *Class. Quantum Grav.* **18** 1637
Carvalho F C and Bezerra de Mello E R 2001 *Class. Quantum Grav.* **18** 5455
[6] Bezerra de Mello E R 2002 *J. Math. Phys.* **43** 1018
[7] Bordag M, Kirsten K and Dowker S 1996 *Commun. Math. Phys.* **128** 371
[8] Bezerra de Mello E R, Bezerra V B and Khusnutdinov N R 2001 *J. Math. Phys.* **42** 562
[9] Saharian A A and Setare M R 2003 *Class. Quantum Grav.* **20** 3765
[10] Saharian A A and Setare M R 2003 Casimir densities for two concentric spherical shells in the global monopole spacetime *Preprint* hep-th/0304008
[11] Saharian A A 1987 *Izv. Akad. Nauk. Arm. SSR. Mat.* **22** 166
Saharian A A 1987 *Sov. J. Contemp. Math. Anal.* **22** 70
Saharian A A 1987 *PhD Thesis* Yerevan, Armenia (in Russian)
[12] Saharian A A 2000 *The generalized Abel–Plana formula Applications to Bessel functions and Casimir effect Report* No. IC/2000 *Preprint* hep-th/0002239
[13] Bender C M and Hays P 1976 *Phys. Rev.* D **14** 2622
[14] Milton K A 1980 *Phys. Rev.* D **22** 1444
[15] Milton K A 1981 *Phys. Lett.* B **104** 49
[16] Olaussen K and Ravndal F 1981 *Phys. Lett.* B **100** 497
[17] Milton K A 1983 *Ann. Phys., NY* **150** 432
[18] Baacke J and Igarashi Y 1983 *Phys. Rev.* D **27** 460
[19] Blau S K, Wisser M and Wipf A 1988 *Nucl. Phys.* B **310** 163
[20] Elizalde E, Bordag M and Kirsten K 1998 *J. Phys. A: Math. Gen.* **31** 1743
[21] Cognola G, Elizalde E and Kirsten K 2001 *J. Phys. A: Math. Gen.* **34** 7311
[22] Mostepanenko V M and Trunov N N 1997 *The Casimir Effect and Its Applications* (Oxford: Oxford University Press)
[23] Plunien G, Muller B and Greiner W 1986 *Phys. Rep.* **134** 87
[24] Milton K A 2002 *The Casimir Effect: Physical Manifestation of Zero-Point Energy* (Singapore: World Scientific)
[25] Bordag M, Mohidden U and Mostepanenko V M 2001 *Phys. Rep.* **353** 1
[26] Birrell N D and Davies P C W 1982 *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press)
[27] Berestetskii V B, Lifshits E M and Pitaevskii L P 1982 *Quantum Electrodynamics* (Oxford: Pergamon)
[28] Christensen S M 1978 *Phys. Rev.* D **17** 946
[29] Wald R M 1978 *Phys. Rev.* D **17** 1477
[30] Romeo A and Saharian A A 2002 *J. Phys. A: Math. Gen.* **35** 1297
[31] Saharian A A 2001 *Phys. Rev.* D **63** 125007
[32] Romeo A and Saharian A A 2001 *Phys. Rev.* D **63** 105019
[33] Saharian A A 2002 *Class. Quantum Grav.* **19** 5039
[34] Avagyan R M, Saharian A A and Yeranyan A H 2002 *Phys. Rev.* D **66** 085023
[35] Balian R and Duplantier B 1978 *Ann. Phys.* **112** 165
[36] Deutsch D and Candelas P 1979 *Phys. Rev.* D **20** 3063
[37] Kennedy G, Critchley R and Dowker J S 1980 *Ann. Phys., NY* **125** 346
[38] Abramowitz M and Stegun I A 1964 *Handbook of Mathematical Functions* (Washington, DC: National Bureau of Standards)